数理解析研究所講究録 第2099巻 2018年 68-72

An alternative proof of the existence of totally real embeddings of 3-manifolds into $\mathbb{C}^3$

Naohiko Kasuya
Department of Mathematics, Kyoto Sangyo University

1 Introduction

Let M^n be a closed, connected, orientable n-manifold and $f\colon M^n \to \mathbb{C}^n$ be an immersion. A point $p \in M^n$ is said to be a *complex tangent* if $df_p(T_pM^n)$ contains a complex line. By Thom's transversality theorem, the set of complex tangents of a generic immersion $f\colon M^n \to \mathbb{C}^n$ is empty or forms a closed $(n-2)$-dimensional submanifold. Elgindi initiated to study the problem of determining the isotopy classes of knots in S^3 which can be realized as the set of complex tangents of an embedding $S^3 \to \mathbb{C}^3$ [2, 3, 4]. In [11] the aurhor and Takase showed that any link L in a closed oriented 3-manifold M^3 can be realized as the set of complex tangents of an embedding $M^3 \to \mathbb{C}^3$ if and only if the homology class $[L]$ is trivial in $H_1(M^3;\mathbb{Z})$.

An immersion is said to be *totally real* if it has no complex tangent, and when the immersion is embedding, it is called a *totally real embedding.* For totally real embeddings, Gromov [8] and Forstnerič [6] proved the following theorem. This is called the *h-principle* for totally real embeddings.

Theorem 1 (Gromov [8], Forstnerič [6])**.** Let M^n be a closed orientable n-manifold with $n \geq 3$. Then, M^n admits a totally real embedding into $\mathbb{C}^n$ if and only if it admits a totally real immersion into $\mathbb{C}^n$ which is regularly homotopic to an embedding.

As a consequence of this theorem, the following is easily shown.

Corollary 2. Any closed orientable 3-manifold admits a totally real embedding into $\mathbb{C}^3$.

Since the proof relies on the h-principle, however, almost nothing can be analyzed about the obtained totally real embedding. On the other hand, some explicit examples of totally real embedings are known. Ahern and Rudin [1] explicitly constructed a totally real embedding of the 3-sphere into $\mathbb{C}^3$. In this article, using Ahern-Rudin's example, we give a new proof of Corollary 2. The argument of the proof was inspired by the work of

Etnyre and Furukawa [5]. They defined the notion of *braided embeddings* and used it to prove the existence of contact embeddings into the standard contact 5-sphere for some contact 3-manifolds. We show that braided embeddings are also useful for constructing totally real embeddings.

2 Preliminaries

In this section, we introduce Ahern-Rudin's example and the notion of braided embeddings.

In [7] Gromov stated that there exist totally real embeddings of the 3-sphere into $\mathbb{C}^3$, but he did not give the proof there. In order to prove it, Ahern and Rudin [1] constructed the following example.

Example 3 (Ahern-Rudin [1]). Let $P(z,w) = \bar{z}\bar{w}(|w|^2 + i|z|^2)$. We consider the 3-sphere as the unit shpere $S^3 = \{(z,w) \mid |z|^2 + |w|^2 = 1\} \subset \mathbb{C}^2$. Then, the embedding $F : S^3 \to \mathbb{C}^3$ defined by

$$F(z,w) = (z, w, P(z,w))$$

is a totally real embedding.

Next, we explain the definition of braided embeddings. First, we recall branched coverings.

Definition 4. Let M^n and Y^n be n-manifolds. A *d-fold branched covering* is a smooth, proper map $p : M^n \to Y^n$ with critical set $B \subset Y^n$ called the *branch locus*, such that p restricted $M^n - p^{-1}(B)$ is a covering map of degree d, and for each $x \in p^{-1}(B)$ there are local coordinates near x and $p(x)$ such that p is given by $(q,z) \mapsto (q, z^m)$ for some $m \in \mathbb{Z}_{>0}$, where q is a coordinate on D^{n-2} and z is a coordinate on the unit disk in $\mathbb{C}$. The integer m is called the *branching index* of p at x. A d-fold branched covering is called *simple* if the pre-image of any point in Y^n has either d or $d-1$ points.

Etnyre and Furukawa [5] defined the following notion.

Definition 5 (Etnyre-Furukawa [5]). Let M^n and Y^n be n-manifolds. An embedding

$$e : M^n \to Y^n \times D^2$$

is called a *braid about* Y^n if $\pi \circ e : M^n \to Y^n$ is a branched covering, where $\pi : Y^n \times D^2 \to Y^n$ is the first projection. If Y^n is embedded in a $(n+2)$-manifold W^{n+2} with trivial normal bundle, then M^n is also embedded in W^{n+2}. This embedding of M^n into W^{n+2}

is called a *braided embedding*. Moreover, a branched covering $p : M^n \to Y^n$ is said to be *braided about* Y^n if there exists a function $f : M^n \to D^2$ such that

$$e : M^n \to Y^n \times D^2 : x \mapsto (p(x), f(x))$$

is an embedding.

For braided embeddings of 3-manifolds, a theorem due to Hilden, Lozano and Montesinos [10] is known. Using the terminology of [5], their theorem can be stated as follows.

Theorem 6 (Hilden-Lozano-Montesinos [10])**.** Every closed oriented 3-manifold M^3 can be braided about the 3-sphere where the corresponding branched covering is a simple 3-fold branched covering.

3 An alternative proof of Corollary 2

Combining Example 2 with Theorem 6, we can give a very simple proof of Corollary 2.

Proof. In Example 2, F also defines an embedding of $\mathbb{C}^2$ into $\mathbb{C}^3$. Since the normal bundle of the embedding F is trivial, we obtain an embedding $\widetilde{F}$ of a tubular neibourhood $\mathbb{C}^2 \times D^2$ into $\mathbb{C}^3$. We also describe the restricted embedding $S^3 \times D^2 \to \mathbb{C}^3$ by the same symbol $\widetilde{F}$. Then, of course, $\widetilde{F}(S^3 \times \{(0,0)\}) = F(S^3)$ is nothing but Ahern-Rudin's example. By Theorem 6, for any closed orientable 3-manifold M^3, there is a function $f : M^3 \to D^2$ such that

$$e : M^3 \to S^3 \times D^2 : x \mapsto (p(x), f(x))$$

is an embedding, where $p : M^3 \to S^3$ is a simple 3-fold branched covering. Since the totally reality is an open condition, for a sufficiently small positive number ϵ, the tangent space of the image of the embedding

$$e_\epsilon : M^3 \to S^3 \times D^2 : x \mapsto (p(x), \epsilon f(x))$$

is close enough to that of $S^3 \times \{(0,0)\}$ in the sense of C^∞-topology, so that the composition with the embedding $\widetilde{F} : S^3 \times D^2 \to \mathbb{C}^3$ is a totally real embedding. Thus, we obtained a totally real embedding $\widetilde{F} \circ e_\epsilon : M^3 \to \mathbb{C}^3$. $\square$

Although the above proof is not by an explicit construction in the sense that the function f is not explicitly given, further analysis of the obtained totally real embedding can be expected because we avoided using the h-principle. For example, it might be easy to take a Seifert surface of the totally real submanifold, since the corresponding branched covering carries informations of the totally real embedding. However, there is a problem. The embedding $\widetilde{F} \circ e_\epsilon : M^3 \to \mathbb{R}^6$ arises from an embedding of M^3 into $\mathbb{R}^5$. Hence, intersting examples like Haefliger knots [9] never appears. In order to realize Haefliger knots as totally real submanifolds explicitly, we need to study braided immersions or the 3-codimensional version of braided embeddings of 3-manifolds.

Problem 7. Can a Haefliger knot be realized as a 3-codimensional braided embedding of the 3-sphre?

The author suspect that Takase's works on Haefliger knots [12, 13] are the keys to approaching this problem. Anyway this is a future problem.

References

[1] Patrick Ahern and Walter Rudin, *Totally real embeddings of S^3 in $\mathbf{C}^3$*, Proc. Amer. Math. Soc. **94** (1985) 460–462.

[2] Ali M. Elgindi. *On the topological structure of complex tangencies to embeddings of S^3 into $\mathbf{C^3}$*, New York J. Math. **18** (2012) 295–313.

[3] Ali M. Elgindi, *A topological obstruction to the removal of a degenerate complex tangent and some related homotopy and homology groups*, Internat. J. Math. 26(5):1550025, 16, 2015.

[4] Ali M. Elgindi, *Totally real perturbations and non-degenerate embeddings of S^3*, New York J. Math. **21** (2015) 1283–1293.

[5] John Etnyre and Ryo Furukawa, *Braided embeddings of contact 3-manifolds in the standard contact 5-sphere*, Journal of Topology **10** (2017), 412–446.

[6] Franc Forstnerič, *On totally real embeddings into $\mathbf{C}^n$*, Exposition. Math. **4** (1986), no. 3, 243–255.

[7] M. Gromov, *Convex integration of differential relations*, Math. USSR-Izv. **7** (1973), 329–343.

[8] Mikhael Gromov, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 9, Springer-Verlag, Berlin, 1986.

[9] A. Haefliger, *Knotted $(4k-1)$-spheres in $6k$-space*, Ann. of Math. **75** (1962) 452-466.

[10] Hugh M. Hilden, María Teresa Lozano, and José María Montesinos, *All three-manifolds are pullbacks of a branched covering S^3 to S^3*, Trans. Amer. Math. Soc. **279** (1983) 729–735.

[11] N. Kasuya and M. Takase, *Knots and links of complex tangents*, Trans. Amer. Math. Soc. **370** (2018) 2023–2038.

[12] M. Takase, *A geometric formula for Haefliger knots*, Topology **43** (2004) 1425–1447.

[13] M. Takase, *The Hopf invariant of a Haefliger knot*, Mathematische Zeitschrift **256** (2007) 35–44.

Department of Mathematics
Faculty of Science, Kyoto Sangyo University
Motoyama, Kamigamo, Kita-ku
Kyoto 603-8555 JAPAN
E-mail address: nkasuya@cc.kyoto-su.ac.jp

京都産業大学理学部数理科学科　粕谷　直彦